# Criteria of spreading and vanishing for a free boundary problem in mathematical ecology

Yuki Kaneko

Department of Pure and Applied Mathematics, Waseda Univ.
JSPS Research Fellow

Joint work with Professor Yoshio Yamada (Waseda Univ.)

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# Mathematical model for spread of species

Find a solution $(u(t,r), h(t))$ of the free boundary problem:

$$
\text{(P)} \quad \begin{cases} u_t - d\Delta u = f(u), & t > 0,\ 0 < r < h(t), \\ u_r(t,0) = 0,\ u(t,h(t)) = 0, & t > 0, \\ h'(t) = -\mu u_r(t,h(t)), & t > 0, \\ h(0) = h_0,\ u(0,r) = u_0(r), & 0 \le r \le h_0. \end{cases}
$$

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$r = |x|\ (x \in \mathbb{R}^N),\ N \ge 2$

$\Delta u = u_{rr} + ((N-1)/r)u_r$

$d,\ \mu,\ h_0$ : positive numbers

$u(t,r)$ : population density

$(0, h(t))$ : habitat of the species

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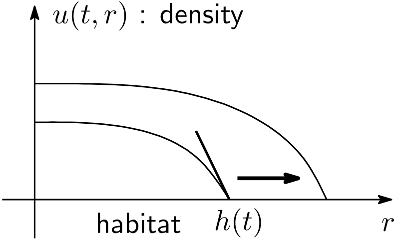



$$u_0 \in C^2(0,h_0),\ u_0'(0) = u_0(h_0) = 0,\ u_0 > 0 \text{ in } [0,h_0).$$

# Basic results for $N = 1$

- Global existence and uniqueness of classical solutions

$$0 < u(t,x) \leq C_1 \quad (t > 0,\ 0 < x < h(t)), \quad 0 < h'(t) \leq \mu C_1 \quad (t > 0).$$

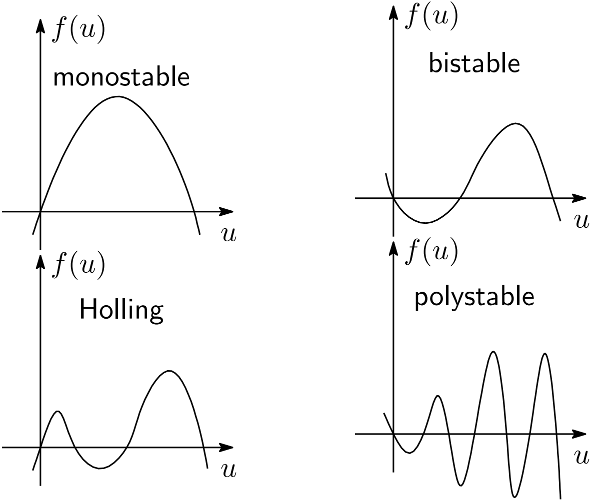
**Asymptotic behavior of solutions (Du-Lin, 2010)**

Let $f(u) = u(a - bu)$ $(a, b > 0)$. Either (i) or (ii) holds true as $t \to \infty$.
(i) Spreading: $\lim_{t\to\infty} h(t) = \infty$, $\lim_{t\to\infty} u(t,x) = a/b$ : locally uniformly.
(ii) Vanishing: $\lim_{t\to\infty} h(t) \leq (\pi/2)\sqrt{d/a}$, $\lim_{t\to\infty} \|u(t,\cdot)\|_{C(0,h(t))} = 0$.

# Purpose

- Consider more general nonlinearity in radially symmetric setting

$$f \in C^1[0,\infty),\ f(0) = f(1) = 0,\ f(u) < 0\ (u > 1),\ f'(0) \neq 0$$



# A Free boundary problem with double fronts

Find a solution $(u(t,r), g(t), h(t))$ of the free boundary problem:

$$
\text{(DFP)} \quad \begin{cases} u_t - d\Delta u = f(u), & t>0,\ g(t)<r<h(t), \\ u(t,g(t))=0,\ u(t,h(t))=0 & t>0, \\ g'(t) = -\mu u_r(t,g(t)), & t>0, \\ h'(t) = -\mu u_r(t,g(t)), & t>0, \\ g(0)=g_0,\ h(0)=h_0,\ u(0,r)=u_0(r), & g_0 \le r \le h_0. \end{cases}
$$

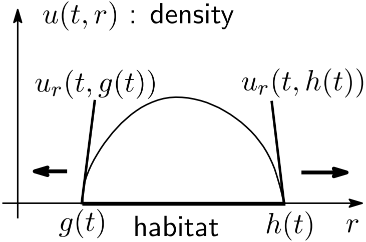
$$r = |x| \ (x \in \mathbb{R}^N),\ N \ge 2$$

$$\Delta u = u_{rr} + ((N-1)/r)u_r$$

$d,\ \mu,\ g_0,\ h_0$ : positive numbers

$u(t,r)$ : population density

$(g(t), h(t))$ : habitat of the species




$$u_0 \in C^2(0,h_0),\ u_0(g_0)=u_0(h_0)=0,\ u_0>0 \text{ in } (g_0,h_0).$$

# Definition of Weak solutions for (DFP)

## Definition 1

Let $G_T := (0,T) \times G$ for some $T > 0$ and large domain $G$.
A non-negative function $u \in L^\infty(G_T) \cap H^1(G_T)$ is called *a weak solution of (DFP) over $G_T$* when $u(t,r)$ satisfies

$$\iint_{G_T} dr^{N-1}u_r\phi_r - r^{N-1}\alpha(u)\phi_t \; drdt - \int_G r^{N-1}\alpha(\tilde{u}_0)\phi(0,r) \; dr = \iint_{G_T} r^{N-1}f(u)\phi \; drdt$$

for any $\phi \in C^1(G_T)$, $\phi = 0$ on $(\{T\} \times G) \cup ([0,T] \times \partial G)$, where

$$\alpha(u) = \begin{cases} u & \text{if } u > 0, \\ u - d/\mu & \text{if } u \leq 0, \end{cases} \quad \tilde{u}_0 = \begin{cases} u_0, & r \in \Omega_0, \\ 0 & r \in G \setminus \Omega_0. \end{cases}$$

# Existence and uniqueness of global solutions

## Theorem (Global weak solution)

There exists a unique weak solution of (DFP) over $G_T$ for any $T > 0$.

## Theorem (Local/Global classical solution)

(DFP) has a unique classical solution $(u, g, h)$ in $(0, T^*]$ for some constant $T^* = T^*(u_0, g_0, h_0) > 0$:

$$0 < u(t,r) \leq C_1 \quad \text{for} \quad 0 < t \leq T^*,\ g_0 < r < h_0,$$
$$-\mu C_2 \leq g'(t) < 0 < h'(t) \leq \mu C_2 \quad \text{for} \quad 0 < t \leq T^*,$$

where $C_1$, $C_2$ depend on $\|u_0\|_{C(g_0,h_0)}$, $\|u_0\|_{C^1(g_0,h_0)}$, respectively.

- Classical solutions satisfy the condition of weak solutions
- Weak solutions, with some properties on smoothness, become classical solutions.