Local well-posedness of the fluid-rigid body interaction problem for compressible fluids

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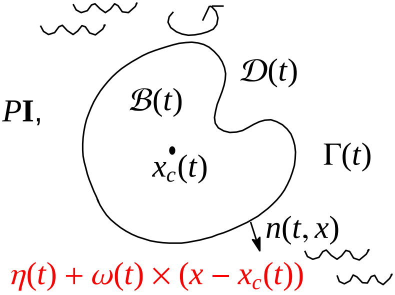
compressible fluid flow

$\mathcal{D}(t) \subset \mathbf{R}^3$: exterior domain, $J_T = (0, T)$.

$$\begin{cases} \rho_t + \mathrm{div}\,(\rho v) = 0 & \text{in } \mathcal{D}(t), t \in J_T \\ \rho(v_t + (v \cdot \nabla)v) - \mathrm{Div}\,\mathbf{T}(v, P) = 0 & \text{in } \mathcal{D}(t), t \in J_T, \\ v(t, x) = h(t, x) & \text{on } \Gamma(t), t \in J_T, \\ v(0) = v_0, \quad \rho(0) = \rho_0 & \text{in } \mathcal{D}\ (:= \mathcal{D}(0)). \end{cases}$$

- $v = (v_1, v_2, v_3)$: velocity field.
- ρ : density.
- $\mathbf{T}(v, P) = 2\mu\mathcal{E}^{(v)} + (\mu' - \mu)\mathrm{div}\, v\mathbf{I} - P\mathbf{I}$,
  $\mathcal{E}^{(v)} = \{(\nabla v) + (\nabla v)^T\}/2$,
  $2\mu + 3\mu' > 0$,
  $\mathbf{I}$: 3×3 identity matrix.
- $P = P(\rho)$: pressure,
  $P \in C^\infty(\mathbf{R}_+)$, $P'(\rho) > 0 \ \ \forall \rho > 0$.
  e.g. $P(\rho) = \rho^\gamma$ $(\gamma > 0)$.



motion of rigid body

$$
\begin{cases}
m\eta'(t) - \int_{\Gamma(t)} \mathbf{T}(v,P)n(t,x)d\sigma = \mathbf{F}(t) & t \in J_T, \\
(J\omega)'(t) - \int_{\Gamma(t)} (x - x_c(t)) \times \mathbf{T}(v,P)n(t,x)d\sigma = \mathbf{M}(t) & t \in J_T, \\
\eta(0) = \eta_0, \quad \omega(0) = \omega_0.
\end{cases}
$$

- $\eta = \eta(t)$: translational velocity.
- $\omega = \omega(t)$: angular velocity.
- $x_c(t)$:center of mass, $x_c(0) = 0$, $n(t,x)$: unit outer normal to $\Gamma(t)$.
- $m > 0$: mass of the rigid body (const.).
- $J(t)$: inertia tensor.

$$
J(t)a \cdot b = \frac{|\mathcal{B}|}{m} \int_{\mathcal{B}} (a \times Q(t)y) \cdot (b \times Q(t)y)dy, \quad \forall a,b \in \mathbf{R}^3,
$$

  where $\mathcal{B} = \mathcal{B}(0)$ and $Q(t)$ is rotation matrix satisfying $Q(0) = \mathbf{I}$.
- $\mathbf{F}$: external force, $\mathbf{M}$: torque.

fluid-rigid body interaction problem

$$
\text{(P)} \quad \begin{cases}
\rho_t + \operatorname{div}(\rho v) = 0 & \text{in } \mathcal{D}(t), t \in J_T, \\
\rho(v_t + (v \cdot \nabla)v) - \operatorname{Div} \mathbf{T}(v, P) = 0 & \text{in } \mathcal{D}(t), t \in J_T, \\
v(t,x) = \eta(t) + \omega(t) \times (x - x_c(t)) & \text{on } \Gamma(t), t \in J_T, \\
v(0) = v_0, \ \ \rho(0) = \rho_0 & \text{in } \mathcal{D}, \\
m\eta'(t) - \int_{\Gamma(t)} \mathbf{T}(v,P)n(t,x)d\sigma = \mathbf{F}(t) & t \in \ J_T, \\
(J\omega)'(t) - \int_{\Gamma(t)} (x - x_c(t)) \times \mathbf{T}(v,P)n(t,x)d\sigma = \mathbf{M}(t) & t \in \ J_T, \\
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\end{cases}
$$

- unknowns : ρ, v, η, ω

Aim

Local in time unique existence theorem in L_p-L_q framework to (P)

known results (strong solution)

incompressible, in L^2 framework

- Galdi and Silvestre ('02) : local, 3D
- Takahashi and Tucsnak ('04) : global, 2D, rigid body is a disk
- Takahashi ('03), Cumsille and Tucsnak ('06),
  Cumsille and Takahashi ('08) : local and global, 2D and 3D

incompressible, in L^p-L^q framework

- Geissert, Götze and Hieber ('13), Götze ('12): local, 2D and 3D

compressible, in L^2 framework

- Boulakia and Guerrero ('09) : local and global, 3D
  initial data : $\rho_0 - \bar{\rho}, v_0 \in H^3(\mathcal{D})$. ($\bar{\rho} > 0$: the mean-value of ρ_0)
  compatibility condition : $\partial_t^i v|_{t=0} = \partial_t^i(\eta + \omega \times (x - x_c))|_{t=0} (i = 0, 1)$.

compressible, in L^p-L^q framework

- Hieber and M. ('15) : local, 3D
  initial data : $\rho_0 - \bar{\rho} \in W^{1,q}(\mathcal{D})$, $v_0 \in (L^q(\mathcal{D}), W^{2,q}(\mathcal{D}))_{1-1/p,p}$.
  compatibility condition : $v_0 = (\eta + \omega \times (x - x_c))|_{t=0} = \eta_0 + \omega_0 \times x$.

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  compatibility condition : $v_0 = (\eta + \omega \times (x - x_c))|_{t=0} = \eta_0 + \omega_0 \times x$.

main theorem

Let $1 < p < \infty$, $3 < q < \infty$. Let $T_0 > 0$ and $\mathbf{F}, \mathbf{M} \in L^p(J_{T_0}; \mathbf{R}^3)$ and $\mathcal{D}$ be an exterior domain of class $C^{2,1}$.
Assume that

$$\rho_0 - \bar{\rho} \in W^{1,q}(\mathcal{D}), v_0 \in (L^q(\mathcal{D}), W^{2,q}(\mathcal{D}))_{1-1/p,p} \text{ and } \eta_0, \omega_0 \in \mathbf{R}^3$$

are satisfying the compatibility condition

$$v_0 = \eta_0 + \omega_0 \times x.$$

Then, there exists $T \in (0, T_0]$ such that problem (P) admits a unique strong solution ρ, v, η and ω on J_T with

$$\begin{aligned}
&\rho \in W^{1,p}(J_T; L^q(\mathcal{D}(\cdot))) \cap L^p(J_T; W^{1,q}(\mathcal{D}(\cdot))),\\
&v \in W^{1,p}(J_T; L^q(\mathcal{D}(\cdot))^3) \cap L^p(J_T; W^{2,q}(\mathcal{D}(\cdot))^3),\\
&\eta, \omega \in W^{1,p}(J_T; \mathbf{R}^3),
\end{aligned}$$

where $\bar{\rho}$ is positive constant describing the mean-value of ρ_0.

sketch of proof

- coordinate transformation
  moving → fixed (by Inoue and Wakimoto '77),
  Euler → Lagrange
- maximal L^p-L^q regularity to linearized problem :

$$
\begin{cases}
\rho_t + \bar{\rho}\,\mathrm{div}\, v = f_0 & \text{in } \mathcal{D}, t \in J_T, \\
v_t - \mathrm{Div}\, \mathbf{T}(v, \rho) = f_1 & \text{in } \mathcal{D}, t \in J_T, \\
v = 0 & \text{on } \Gamma, t \in J_T, \\
v(0) = v_0, \ \ \rho(0) = \rho_0 - \bar{\rho} & \text{in } \mathcal{D}, \\
m\eta' - \int_\Gamma \mathbf{T}(v, \rho) n d\sigma = g_0 & t \in J_T, \\
J\omega' - \int_\Gamma \xi \times \mathbf{T}(v, \rho) n d\sigma = g_1 & t \in J_T, \\
\eta(0) = \eta_0, \ \ \omega(0) = \omega_0. &
\end{cases}
$$

$\xi \in \mathcal{D}$: Lagrange coordinates.

- contraction mapping principle

similar result : $\mathcal{B}(t)$, $\mathcal{D}(t) \subset$ bdd domain $\mathcal{O}$

(P')

$$
\begin{cases}
\rho_t + \mathrm{div}\,(\rho v) = 0 & \text{in } \mathcal{D}(t), t \in J_T,\\
\rho(v_t + (v\cdot\nabla)v) - \mathrm{Div}\,\mathbf{T}(v,P) = 0 & \text{in } \mathcal{D}(t), t \in J_T,\\
v(t,x) = \eta(t) + \omega(t)\times(x - x_c(t)) & \text{on } \Gamma(t), t \in J_T,\\
v(t,x) = 0 & \text{on } \partial\mathcal{O}, t \in J_T,\\
v(0) = v_0, \quad \rho(0) = \rho_0 & \text{in } \mathcal{D},\\
m\eta'(t) - \int_{\Gamma(t)} \mathbf{T}(v,P)n(t,x)d\sigma = \mathbf{F}(t) & t \in J_T,\\
(J\omega)'(t) - \int_{\Gamma(t)}(x - x_c(t))\times\mathbf{T}(v,P)n(t,x)d\sigma = \mathbf{M}(t) & t \in J_T,\\
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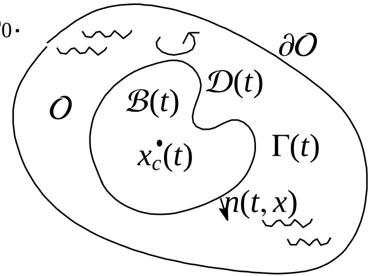

similar result

Let $1 < p < \infty$ and $3 < q < \infty$. Let $T_0 > 0$ and $F, M \in L^p(J_{T_0}; \mathbf{R}^3)$. Let $\mathcal{O} \subset \mathbf{R}^3$ be a bounded domain of class $C^{2,1}$ and $\mathcal{B} \subset \mathcal{O}$. Assume that

$$\rho_0 - \overline{\rho} \in W^{1,q}(\mathcal{D}), v_0 \in (L^q(\mathcal{D}), W^{2,q}(\mathcal{D}))_{1-1/p,p} \text{ and } \eta_0, \omega_0 \in \mathbf{R}^3$$

are satisfying the compatibility condition $v_0 = \eta_0 + \omega_0 \times x$. If

$$\mathrm{dist}(\mathcal{B}, \partial\mathcal{O}) > d \quad \text{for some } d > 0,$$

then there exists $T \in (0, T_0]$ such that the system (P') admits a unique, strong solution (ρ, v, η, ω) on $J_T = (0, T)$ within the class

$$\begin{aligned}
&\rho \in W^{1,p}(J_T; L^q(\mathcal{D}(\cdot))) \cap L^p(J_T; W^{1,q}(\mathcal{D}(\cdot))),\\
&v \in W^{1,p}(J_T; L^q(\mathcal{D}(\cdot))^3) \cap L^p(J_T; W^{2,q}(\mathcal{D}(\cdot))^3),\\
&\eta, \omega \in W^{1,p}(J_T; \mathbf{R}^3).
\end{aligned}$$