The initial-boundary value problems for scalar conservation laws with stochastic forcing

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Our problem

Scalar Conservation Laws with stochastic perturbation

$$
(1) \quad \begin{cases} du + \mathrm{div}(A(u))dt = g(u)d\beta(t) & \text{in } (0,T) \times D \\ u(0,\cdot) = u_0(\cdot) & \text{on } D \\ u \cong u_b & \text{on } (0,T) \times \partial D \end{cases}
$$

Here,

- $T > 0$ and $d \in \mathbb{N}$,
- $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t \geq 0}, \mathbb{P})$ is a probability space with a filtration,
- $D \subset \mathbb{R}^d$ is a bounded convex domain with a smooth boundary ∂D,
- A (called Flux function) is a function from $\mathbb{R}$ to $\mathbb{R}^d$,
- g is a function from $\mathbb{R}$ to $\mathbb{R}$,
- β is a Brownian motion with respect to $\{\mathscr{F}_t\}_{t \geq 0}$.

Deterministic case

(Determinisitc) Scalar Conservation Laws

$$(2) \quad \begin{cases} \partial_t u + \mathrm{div}(A(u)) = 0 & \text{in } (0,T) \times D \\ u(0,\cdot) = u_0(\cdot) & \text{on } D \\ u \cong u_b & \text{on } (0,T) \times \partial D \end{cases}$$

Problems:

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Problems:

- We cannot get the existence of a strong solution and the uniqueness of weak solutions in general.

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Problems:

- We cannot get the existence of a strong solution and the uniqueness of weak solutions in general.
- It is well known that a solution is constant along characteristic curves.

Entropy solution and Kinetic solution

Entropy solution

For all convex $S \in C^\infty(\mathbb{R})$ and $\varphi \in C_c^\infty([0,T) \times D)$ with $\varphi \geq 0$,

$$\int_0^T \int_D \Big(S(u)\partial_t \varphi + \eta(u) \cdot \nabla \varphi \Big) + \int_D S(u_0)\varphi + \int_0^T \int_{\partial D} S(u_b)\varphi \geq 0,$$

where $\eta(\xi) := S'(\xi)a(\xi)$, $a := A'$.

Kinetic solution

$\exists m \in \mathcal{M}_+^b([0,T) \times D \times \mathbb{R})$, $\exists \bar{m}_\pm \in L^1_{loc}((0,T) \times \partial D \times \mathbb{R})$:nonnegative s.t. $\forall \varphi \in C_c^\infty([0,T) \times \overline{D} \times \mathbb{R})$,

$$\int_0^T \int_D \int_{\mathbb{R}} f_\pm (\partial_t + a(\xi) \cdot \nabla)\varphi + \int_D \int_{\mathbb{R}} f_\pm^0 \varphi + M \int_0^T \int_{\partial D} \int_{\mathbb{R}} f_\pm^b \varphi$$

$$= \int_{[0,T) \times D \times \mathbb{R}} \partial_\xi \varphi \; dm + \int_{[0,T) \times \partial D \times \mathbb{R}} \partial_\xi \varphi \; d\bar{m}_\pm.$$

where M is constant depending on flux A, $\quad f_+ = \mathbf{1}_{u > \xi}$ and $f_- = 1 - f_+$.

Known results (deterministic case)

Bardos, Le Roux, Nédélec (1979)

- They proved that if the initial datum u_0 is $BV(D)$ and the boundary datum is C^2-regular, there exists a unique entropy solution.

Otto (1996)

- In the case of L^∞ data, he proved the existence and the uniqueness of entropy solution.

Imbert, Vovelle (2004)

- In the case of L^∞ data, they proved the equivalence between entropy solutions and kinetic solutions.

Known results (stochastic case)

Debussche, Vovelle (2010)

- They proved that there exists a unique kinetic solution on periodic domain.

Bauzet, Vallet, Wittobold (2013)

- Under the assumption that the flux function A is globally Lipschitz continuous, they proved that there exists a unique entropy solution with 0-Dirichlet boundary condition.

Stochastic inviscid Burgers' equation

$$du + \partial_x\left(\frac{u^2}{2}\right)dt = g(x,u)d\beta(t)$$

Crucial difference between deterministic case and stochastic case

In the deterministic case, we can get L^∞ a priori estimate:

$$||u(t)||_{L^\infty} \leq C.$$

But in the stochastic case, we can only get L^p a priori estimate:

$$||u(t)||_{L^p} \leq C_p.$$

Motivation

- Can we prove the uniqueness and the existence of stochastic kinetic solution in the case of bounded domains?

- Can we relax the assumption of global Lipschitz continuity of the flux function A?

Assumption

Scalar Conservation Laws with stochastic perturbation

$$
(1) \quad \begin{cases} du + \mathrm{div}(A(u))dt = g(u)d\beta(t) & \text{in } (0,T)\times D \\ u(0,\cdot) = u_0(\cdot) & \text{on } D \\ u \cong u_b & \text{on } (0,T)\times \partial D \end{cases}
$$

1. The flux function A is of class $C^2(\mathbb{R} : \mathbb{R}^d)$ and its derivatives have at most polynomial growth.
2. The function $g : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous.
3. $u_0 \in L^\infty(D)$ and $u_b \in L^\infty((0,T)\times \partial D)$.

Definition(kinetic solution)

A measurable function $u : \Omega \times (0,T) \times D \to \mathbb{R}$ is called a kinetic solution to (1) if

$\exists m \in \mathcal{M}_+^b([0,T) \times D \times (-N,N))$,
$\exists \bar{m}_\pm \in L^1_{loc}((0,T) \times \partial D \times (-N,N))$: nonnegative
s.t. $\forall \varphi \in C_c^\infty([0,T) \times \overline{D} \times (-N,N))$,

$$\int_0^T \int_D \int_{-N}^N f_\pm (\partial_t + a \cdot \nabla)\varphi + \int_D \int_{-N}^N f_\pm^0 \varphi(0) + M_N \int_0^T \int_{\partial D} \int_{-N}^N f_\pm^b \varphi$$
$$= - \int_0^T \int_D g(u)\,\varphi(t,x,u)\ dx d\beta(t) - \frac{1}{2} \int_0^T \int_D g^2(u)\,\partial_\xi \varphi(t,x,u)\ dxdt$$
$$+ \int_{[0,T)\times D \times (-N,N)} \partial_\xi \varphi\ dm + \int_0^T \int_{\partial D} \int_{-N}^N \partial_\xi \varphi\ d\bar{m}_N^\pm \qquad \text{a.s.},$$

where, $\quad M_N = \max_{|\xi| \le N} |a(\xi)|, \quad f_+ = \mathbf{1}_{u>\xi}$ and $f_- = f_+ - 1$.

Main result

Theorem

We assume $A'' \in L^\infty(\mathbb{R})$. Then, there exists a unique kinetic solution to (1) with data (u_0, u_b). Moreover, given data (u_0^1, u_b^1) and (u_0^2, u_b^2), the L^1-contruction property holds:

$$\mathbb{E}\left|\left|u^1(t) - u^2(t)\right|\right|_{L^1} \leq \mathbb{E}\left|\left|u_0^1 - u_0^2\right|\right|_{L^1} + M\left|\left|u_b^1 - u_b^2\right|\right|_{L^1},$$

where, M is a constant depending on $||u_{1,b}||_{L^\infty}$ and $||u_{2,b}||_{L^\infty}$.

Thank you for your attention!