The initial-boundary value problems for scalar conservation laws with stochastic forcing

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Our problem

Scalar Conservation Laws with stochastic perturbation

\[
\begin{aligned}
&du + \text{div}(A(u))dt = g(u)d\beta(t) \quad \text{in } (0, T) \times D \\
&u(0, \cdot) = u_0(\cdot) \quad \text{on } D \\
&u \equiv u_b \quad \text{on } (0, T) \times \partial D
\end{aligned}
\]

Here,

- \( T > 0 \) and \( d \in \mathbb{N} \),
- \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) is a probability space with a filtration,
- \( D \subset \mathbb{R}^d \) is a bounded convex domain with a smooth boundary \( \partial D \),
- \( A \) (called Flux function) is a function from \( \mathbb{R} \) to \( \mathbb{R}^d \),
- \( g \) is a function from \( \mathbb{R} \) to \( \mathbb{R} \),
- \( \beta \) is a Brownian motion with respect to \( \{\mathcal{F}_t\}_{t \geq 0} \).
(Deterministic) Scalar Conservation Laws

\[
\begin{aligned}
\partial_t u + \text{div}(A(u)) &= 0 & \text{in} & (0, T) \times D \\
u(0, \cdot) &= u_0(\cdot) & \text{on} & D \\
u \cong u_b & \text{on} (0, T) \times \partial D
\end{aligned}
\] (2)

Problems:
Deterministic case

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Problems:

- We cannot get the existence of a strong solution and the uniqueness of weak solutions in general.
Deterministic case

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\end{align*}
\]

Problems:

- We cannot get the existence of a strong solution and the uniqueness of weak solutions in general.
- It is well known that a solution is constant along characteristic curves.
Entropy solution

For all convex $S \in C^\infty(\mathbb{R})$ and $\varphi \in C_c^\infty([0, T) \times D)$ with $\varphi \geq 0$,

$$\int_0^T \int_D \left( S(u) \partial_t \varphi + \eta(u) \cdot \nabla \varphi \right) + \int_D S(u_0) \varphi + \int_0^T \int_{\partial D} S(u_b) \varphi \geq 0,$$

where $\eta(\xi) := S'(\xi) a(\xi)$, $a := A'$.

Kinetic solution

$$\exists m \in M^b_+(\mathbb{R}) \times D \times \mathbb{R}, \exists \bar{m}_\pm \in L^1_{loc}((0, T) \times \partial D \times \mathbb{R}); \text{nonnegative s.t.} \ \forall \varphi \in C_c^\infty([0, T) \times \bar{D} \times \mathbb{R}),$$

$$\int_0^T \int_D \int_{\mathbb{R}} f_\pm(\partial_t + a(\xi) \cdot \nabla) \varphi + \int_D \int_{\mathbb{R}} f^0_\pm \varphi + M \int_0^T \int_{\partial D} \int_{\mathbb{R}} f^b_\pm \varphi$$

$$= \int_{[0, T) \times D \times \mathbb{R}} \partial_\xi \varphi \ dm + \int_{[0, T) \times \partial D \times \mathbb{R}} \partial_\xi \varphi \ d\bar{m}_\pm.$$

where $M$ is constant depending on flux $A$, $f_+ = 1_{u > \xi}$ and $f_- = 1 - f_+$. 
Known results (deterministic case)

Bardos, Le Roux, Nédélec (1979)
- They proved that if the initial datum $u_0$ is $BV(D)$ and the boundary datum is $C^2$-regular, there exists a unique entropy solution.

Otto (1996)
- In the case of $L^\infty$ data, he proved the existence and the uniqueness of entropy solution.

Imbert, Vovelle (2004)
- In the case of $L^\infty$ data, they proved the equivalence between entropy solutions and kinetic solutions.
Known results (stochastic case)

Debussche, Vovelle (2010)

- They proved that there exists a unique kinetic solution on periodic domain.

Bauzet, Vallet, Wittobold (2013)

- Under the assumption that the flux function $A$ is globally Lipschitz continuous, they proved that there exists a unique entropy solution with 0-Dirichlet boundary condition.

Stochastic inviscid Burgers’ equation

$$du + \partial_x \left( \frac{u^2}{2} \right) dt = g(x, u) d\beta(t)$$
Crucial difference between deterministic case and stochastic case

In the deterministic case, we can get $L^\infty$ a priori estimate:

$$\|u(t)\|_{L^\infty} \leq C.$$ 

But in the stochastic case, we can only get $L^p$ a priori estimate:

$$\|u(t)\|_{L^p} \leq C_p.$$
Motivation

- Can we prove the uniqueness and the existence of stochastic kinetic solution in the case of bounded domains?
- Can we relax the assumption of global Lipschitz continuity of the flux function $A$?
Assumption

Scalar Conservation Laws with stochastic perturbation

\[
\begin{aligned}
\frac{du}{dt} + \text{div}(A(u))dt &= g(u)d\beta(t) \quad \text{in } (0, T) \times D \\
\quad u(0, \cdot) &= u_0(\cdot) \quad \text{on } D \\
\quad u &\equiv u_b \quad \text{on } (0, T) \times \partial D
\end{aligned}
\]

1. The flux function $A$ is of class $C^2(\mathbb{R} : \mathbb{R}^d)$ and its derivatives have at most polynomial growth.

2. The function $g : \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz continuous.

3. $u_0 \in L^\infty(D)$ and $u_b \in L^\infty((0, T) \times \partial D)$. 
Definition (kinetic solution)

A measurable function $u : \Omega \times (0, T) \times D \rightarrow \mathbb{R}$ is called a kinetic solution to (1) if

\[
\exists m \in \mathcal{M}_b^b([0, T) \times D \times (-N, N)),
\]

\[
\exists \bar{m}_\pm \in L^1_{\text{loc}}((0, T) \times \partial D \times (-N, N)) : \text{nonnegative}
\]

\[s.t.\ \forall \varphi \in C^\infty_c([0, T) \times \bar{D} \times (-N, N)),\]

\[
\int_0^T \int_D \int_{-N}^N f_\pm (\partial_t + a \cdot \nabla) \varphi + \int_D \int_{-N}^N f^0_\pm \varphi(0) + M_N \int_0^T \int_{\partial D} \int_{-N}^N f^b_\pm \varphi
\]

\[= -\int_0^T \int_D g(u) \varphi(t, x, u) \, dx \, dB(t) - \frac{1}{2} \int_0^T \int_D g^2(u) \partial_\xi \varphi(t, x, u) \, dx \, dt
\]

\[+ \int_{[0, T) \times D \times (-N, N)} \partial_\xi \varphi \, dm + \int_0^T \int_{\partial D} \int_{-N}^N \partial_\xi \varphi \, d\bar{m}_\pm^N \quad \text{a.s.,}
\]

where, $M_N = \max_{|\xi| \leq N} |a(\xi)|$, $f_+ = 1_{u > \xi}$ and $f_- = f_+ - 1$. 
Main result

Theorem

We assume $A'' \in L^\infty(\mathbb{R})$. Then, there exists a unique kinetic solution to (1) with data $(u_0, u_b)$. Moreover, given data $(u_{10}, u_{1b})$ and $(u_{20}, u_{2b})$, the $L^1$-contraction property holds:

$$
\mathbb{E} \left| \left| u^1(t) - u^2(t) \right| \right|_{L^1} \leq \mathbb{E} \left| \left| u_{0}^1 - u_{0}^2 \right| \right|_{L^1} + M \left| \left| u_{b}^1 - u_{b}^2 \right| \right|_{L^1},
$$

where, $M$ is a constant depending on $\left| \left| u_{1,b} \right| \right|_{L^\infty}$ and $\left| \left| u_{2,b} \right| \right|_{L^\infty}$.
Thank you for your attention!