Extension criterion on time local solutions to the Navier-Stokes equations

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Navier-Stokes Equation (time-depending incompressible viscous flow)

\[ u_t - \Delta u + (u, \nabla)u + \nabla p = 0 \quad x \in \mathbb{R}^n, \ t > 0 \]
\[ \text{div} \ u = 0 \quad x \in \mathbb{R}^n, \ t > 0 \]
\[ u(x, 0) = u_0(x) \quad x \in \mathbb{R}^n \] (0.1)

Given functions
\[ u_0(x) : \text{initial velocity of the fluid at a point} \ x \]

Unknown functions
\[ u(x, t) : \text{velocity of the fluid at} \ (x, t) \]
\[ p(x, t) : \text{pressure of the fluid at} \ (x, t) \]

This Cauchy problem is time \textit{locally} well-posed.
Known results: Extension Criteria for strong solutions

$u$ can be continued after $t = T$ under following assumptions

  \[ \int_0^T \| \omega(t) \|_{\dot{B}^0_{\infty, \infty}} dt < \infty \]

- Giga (1986) \quad $u \in L^s(0, T; L^r)$ for \( \frac{2}{s} + \frac{n}{r} = 1 \) with \( n < r \leq \infty \)

  \[ \int_0^T \| u(t) \|_{F^{-\alpha}_{\infty, \infty}} dt < \infty \quad \text{for all} \quad 0 < \alpha < 1 \quad \text{with} \quad \gamma = \frac{2}{1 - \alpha} \]

  \[ \int_{\epsilon}^T \| \tilde{\omega}(t) \|_{BMO} dt < \infty \quad \text{for some} \quad 0 < \epsilon < T \]

where $\tilde{\omega} = (\omega_1, \omega_2, 0)$,
We consider the following Cauchy problem for incompressible Navier-Stokes equation.

\[ u_t - \Delta u + (u, \nabla)u + \nabla p = 0 \quad \mathbb{R}^n \times (0, T) \]
\[ \text{div} u = 0 \quad \mathbb{R}^n \times (0, T) \]
\[ u(x, 0) = u_0(x) \quad \mathbb{R}^n \]

(0.2)

where \( n = 3, 4, \) \( u_0 \in L^2_0(\mathbb{R}^n). \) We work on classical solutions which satisfy

\[ u \in L^\infty(0, T; L^2(\mathbb{R}^n)) \cap L^2(0, T; H^1(\mathbb{R}^n)) \]

and discuss the partial regularity at the final time \( T \) and time-extension criterion.
Theorem 1.1

There is a positive number $\epsilon$ satisfying the following property. Assume that for a point $z_0 = (x_0, T) \in \mathbb{R}^n \times T$ the inequality

$$\limsup_{r \to 0} \sup_{T - r^2 \leq t < T} \frac{1}{r^{n-2}} \int_{B(x_0, r)} |u(x, t)|^2 \, dx < \epsilon$$

holds. Then $z_0$ is a regular point.

Here, we say $z_0 = (x_0, T)$ a regular point if

$$\text{for some } r > 0 \quad u \in L^\infty(B(x_0, r) \times (T - r^2, T))$$
Main Theorem: Extension Criterion

Theorem 1.2

Assume that there exists $R > 0$ such that

$$\sup_{0 < r \leq R, x_0 \in \mathbb{R}^n} \sup_{T - r^2 \leq t < T} \frac{1}{r^{n-2}} \int_{B(x_0, r)} |u(x, t)|^2 \, dx < \epsilon$$

holds. Then there exists $T' > T$, and $u$ can be extended to the classical solution on $\mathbb{R}^n \times (0, T')$. Moreover, this solution satisfies

$$u \in L^\infty(0, T'; L^2(\mathbb{R}^n)) \cap L^2(0, T'; H^1(\mathbb{R}^n))$$