Extension criterion on time local solutions to the Navier-Stokes equations

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Introduction

Navier-Stokes Equation (time-depending incompressible viscous flow)

$$
\begin{aligned}
u_t - \Delta u + (u, \nabla)u + \nabla p = 0 & \quad x \in \mathbb{R}^n, t > 0 \\
\mathrm{div}\, u = 0 & \quad x \in \mathbb{R}^n, t > 0 \\
u(x,0) = u_0(x) & \quad x \in \mathbb{R}^n
\end{aligned}
\tag{0.1}
$$

Given functions

$u_0(x)$: initial velocity of the fluid at a point x

Unknown functions

$u(x,t)$: velocity of the fluid at (x,t)

$p(x,t)$: pressure of the fluid at (x,t)

This Cauchy problem is time locally well-posed.

Known results : Extension Criteria for strong solutions

u can be continued after $t = T$ under following assumptions

- Beale-Kato-Majda(1984), Kato-Ponce(1988), Kozono-Ogawa-Taniuchi(2002)

$$\int_0^T \|\omega(t)\|_{\dot{B}^0_{\infty,\infty}} dt < \infty$$

- Giga(1986) $\quad u \in L^s(0, T; L^r) \;\; \textit{for} \;\; \frac{2}{s} + \frac{n}{r} = 1 \;\; \textit{with} \;\; n < r \leq \infty$
- Kozono-Shimada(2004)

$$\int_0^T \|u(t)\|^{\gamma}_{\dot{F}^{-\alpha}_{\infty,\infty}} dt < \infty \;\; \textit{for all } 0 < \alpha < 1 \;\; \textit{with } \gamma = \frac{2}{1-\alpha}$$

- Kozono-Yatsu(2004)

$$\int_\epsilon^T \|\tilde{\omega}(t)\|_{BMO} dt < \infty \;\; \textit{for some } 0 < \epsilon < T$$

where $\tilde{\omega} = (\omega_1, \omega_2, 0)$,

Our Problem

We consider the following Cauchy problem for incompressible Navier-Stokes equation.

$$
\begin{aligned}
u_t - \Delta u + (u, \nabla) u + \nabla p = 0 & \quad \mathbb{R}^n \times (0, T) \\
\mathrm{div} u = 0 & \quad \mathbb{R}^n \times (0, T) \\
u(x, 0) = u_0(x) & \quad \mathbb{R}^n
\end{aligned}
\tag{0.2}
$$

where $n = 3, 4,\ u_0 \in L^2_\sigma(\mathbb{R}^n)$. We work on classical solutions which satisfy

$$u \in L^\infty(0, T; L^2(\mathbb{R}^n)) \cap L^2(0, T; H^1(\mathbb{R}^n))$$

and discuss the partial regularity at the final time T and time-extention criterion.

Partial Regularity

Theorem1.1

There is a positive number ϵ *satisfying the following property. Assume that for a point* $z_0 = (x_0, T) \in \mathbb{R}^n \times T$ *the inequality*

$$\limsup_{r \to 0} \sup_{T - r^2 \leq t < T} \frac{1}{r^{n-2}} \int_{B(x_0, r)} |u(x,t)|^2 dx < \epsilon$$

holds. Then z_0 *is a regular point.*

Here, we say $z_0 = (x_0, T)$ a regular point if

$$\textit{for some } r > 0 \quad u \in L^\infty(B(x_0, r) \times (T - r^2, T))$$

Main Theorem : Extension Criterion

Theorem1.2

Assume that there exists $R > 0$ *such that*

$$\sup_{0<r\leq R, x_0\in\mathbb{R}^n} \sup_{T-r^2\leq t<T} \frac{1}{r^{n-2}} \int_{B(x_0,r)} |u(x,t)|^2 dx < \epsilon$$

holds. Then there exists $T' > T$, *and* u *can be extended to the classical solution on* $\mathbb{R}^n \times (0, T')$. *Moreover, this solution satisfies*

$$u \in L^\infty(0, T'; L^2(\mathbb{R}^n)) \cap L^2(0, T'; H^1(\mathbb{R}^n))$$